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Canonical subgroup matrices for free Abelian groups of finite rank: application to intersection and union subgroups

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Abstract. Given a free Abelian group $\mathbf{T}$ of finite rank, canonical subgroup matrices ${}^{\star}\mathbb{M}$ are defined with respect to an arbitrary but fixed basis of $\mathbf{T}$, which uniquely determine all subgroups $\mathbf{S}$ of $\mathbf{T}$ of finite index. The use of these matrices allows the classification of all subgroups of fixed finite index. An application is made in the construction of the intersection group and the union group of two arbitrary subgroups of finite index and upper and lower bounds on the index of these groups are found. Physical applications occur for rank three in the systematic symmetry analysis of domain structures, which appear in structural phase transitions in solids.

1. Introduction

Three-dimensional crystallographic space groups possess an invariant Abelian subgroup of translations and a study of the lattice of subgroups of space groups requires the systematic investigation of the lattice of subgroups of the translation groups. Such subgroup lattices are of importance in the symmetry analysis of domain structures which appear in structural phase transitions of solids which are accompanied by a symmetry reduction. Thus the space group $\mathcal{H}$ of the distorted phase is a proper subgroup of the space group $\mathcal{G}$ of the parent phase which implies that $\mathcal{H} \subset \mathcal{G}$. Due to the symmetry reduction at the phase transition the distorted phase can appear in several homogenous simultaneously co-existing states which have the same structure but different orientations and/or locations in space when referred to a particular coordinate system. The theoretical background of the symmetry analysis can be found in many papers from which we mention only some, namely Janovec [1–3], Van Tendeloo and Amelinckx [4], Kopsky [5], Zikmund [6]. The existence of software to support domain structure symmetry analysis has been reported recently by Davies *et al* [7, 8]. In these latter papers the role of intersection groups in determining the symmetry groups of domain pairs, and of intermediate groups (related to union groups) in determining minimal permutable sets of domain states is explained and illustrated. Here, the translation groups are free Abelian groups of rank three but the results presented in this paper apply to arbitrary finite rank. For the background material on infinite Abelian groups the reader is referred to the book by Fuchs [9] and similarly for basic concepts on integral matrices and number theory one should consult the book by Newman [10] and that by Rademacher [11], respectively.

2. Subgroups of free Abelian groups

2.1. Canonical subgroup matrices

Let **T** denote a free Abelian group of rank n and let a subgroup **S** of **T** be defined by the $n \times n$ integral matrix $\mathbb{M}$ of positive determinant. The n basis vectors of **S** written as row vectors $\vec{s} := (s_1, s_2, \ldots, s_n)$ are defined as integral linear combinations of the n basis vectors $\vec{t} := (t_1, t_2, \ldots, t_n)$ of **T** by

$$\vec{s} = \vec{t}\,\mathbb{M}. \tag{1}$$

The basis vectors $\vec{t}$ of **T** are arbitrary but once chosen, they are *fixed*. The determinant of $\mathbb{M}$ equals the index of **S** in **T**. With respect to the basis $\vec{t}$, any element of **S** is represented by a column vector which is an unique integral linear combination of the columns of $\mathbb{M}$. The subset of $\mathbb{Z}^n$ generated by the addition of columns of $\mathbb{M}$ over $\mathbb{Z}$ is a group isomorphic to **S** and for convenience we shall identify **S** with this subset. We show that **S** may be defined by a (canonical) triangular integral matrix ${}^{\star}\mathbb{M}$ which is column-equivalent to the original integral matrix $\mathbb{M}$:

$$^{\star}\mathbb{M} = \begin{bmatrix} D_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ E_{n-1,n} & D_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ E_{n-2,n} & E_{n-2,n-1} & D_{n-2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ E_{3,n} & E_{3,n-1} & E_{3,n-2} & \cdots & D_3 & 0 & 0 \\ E_{2,n} & E_{2,n-1} & E_{2,n-2} & \cdots & E_{2,3} & D_2 & 0 \\ E_{1,n} & E_{1,n-1} & E_{1,n-2} & \cdots & E_{1,3} & E_{1,2} & D_1 \end{bmatrix} \tag{2}$$

where $1 \leqslant D_j,\ j = n, n-1, \ldots, 2, 1$, and $0 \leqslant E_{j,i} \leqslant D_j - 1,\ i = n, n-1, \ldots, j+1$, $j = n-1, n-2, \ldots, 2, 1$. For further details, like Smith normal form, which we do not need to invoke here, the reader is referred for instance to the book by Pohst and Zassenhaus [12] or that by Fuchs [9].

To prove this we exploit the Euclidean algorithm to find the greatest common divisor (a, b) of two positive integers a, b. We adopt the convention that $M_{i,j}$ denotes the *current* value in the ith row and jth column of $\mathbb{M}$ as $\mathbb{M}$ is reduced step by step to the column-equivalent triangular form ${}^{\star}\mathbb{M}$. Each step is an elementary column operation of the following types: (i) interchange of two columns, (ii) change of the sign of a column, (iii) adding (subtracting) one column to (from) another column. The sign of the determinant of $\mathbb{M}$ is changed by these operations at most. There is no need to keep track of changes of sign since both the initial $\mathbb{M}$ and the final canonical form ${}^{\star}\mathbb{M}$ have positive determinant.

Consider the first row of $\mathbb{M}$. Let $M_{1,j}$ denote the first non-zero element in the first row for some $j = 1, 2, \ldots, n$. This element must exist since $\det \mathbb{M}$ is not zero. If $j > 1$ then interchange the first and jth columns. If $M_{1,1}$ is less than zero then change the sign of the first column. If there are no more non-zero elements in the first row then it possesses the desired canonical form. Otherwise let $M_{1,k}$ denote the next non-zero element in the first row for some $k = 2, 3, \ldots, n$. If $k > 2$ then interchange the second and kth columns. If $M_{1,2}$ is less than zero then change the sign of the second column. If $M_{1,1} > M_{1,2}$ then interchange the first and second columns. Let $a = M_{1,1}$ and $b = M_{1,2}$. We now apply the Euclidean algorithm to the pair of positive integers a, b where $a < b$ to find the greatest common divisor (a, b) of a and b.

(A) Continually subtract the first column from the second column until either $M_{1,2} = 0$ or $M_{1,1} > M_{1,2}$. In the latter case interchange the first and second columns and repeat (A). Continue until $M_{1,2} = 0$ in which case $(a, b) = M_{1,1}$. The well-ordering principle for the integers guarantees that (a, b) is found after a finite number of steps. Repeat the above for all non-zero elements in the first row to give $M_{1,k} = 0$ for $k = 2, 3, \ldots, n$. The final value of $M_{1,1} = D_1$ is the greatest common divisor of all the non-zero elements in the first row of the initial form of $\mathbb{M}$.

The $n-1$ elements $M_{2,k}$, $k = 2, 3, \ldots, n$ in the second row are treated in the same way to yield $M_{2,2} = D_2$ with $D_2 \geqslant 1$ is the greatest common divisor of the non-zero elements $M_{2,k}$, $k = 2, 3, \ldots, n$. If $M_{2,1} < 0$ or $M_{2,1} \geqslant M_{2,2}$, then the second column is repeatedly added to or subracted from the first column until $0 \leqslant M_{2,1} \leqslant M_{2,2} - 1$, a process which is completed in a finite number of steps as above. The second row is now in the desired canonical form. The above process is repeated for the remaining $n - 2$ rows when the canonical form ${}^\star\mathbb{M}$ is obtained.

2.2. Uniqueness of canonical subgroup matrices

The canonical matrices ${}^\star\mathbb{M}$ are unique in the sense that one and only one canonical matrix defines the subgroup $\mathbf{S}$. We give a detailed proof for the case $n = 3$. Let $\vec{s}$ and $\vec{s}\,'$ denote two bases of $\mathbf{S}$ given by the canonical matrices ${}^\star\mathbb{M}$ and ${}^\star\mathbb{M}'$:

$$ {}^\star\mathbb{M} = \begin{bmatrix} \ell & 0 & 0 \\ x & m & 0 \\ y & z & n \end{bmatrix} \tag{3} $$

$$ {}^\star\mathbb{M}' = \begin{bmatrix} \ell' & 0 & 0 \\ x' & m' & 0 \\ y' & z' & n' \end{bmatrix}. \tag{4} $$

Since $\vec{s}$ and $\vec{s}\,'$ are bases for the same subgroup $\mathbf{S}$, then $\vec{s}\,' = \vec{s}\ \mathbb{A}$ so that ${}^\star\mathbb{M}' = {}^\star\mathbb{M}\ \mathbb{A}$ where in particular $\mathbb{A} \in \mathbb{GL}(3, \mathbb{Z})$ must hold:

$$ \mathbb{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}. \tag{5} $$

It follows immediately from the triangular form of ${}^\star\mathbb{M}$ and ${}^\star\mathbb{M}'$ that $b = c = f = 0$ so that $\mathbb{A}$ is triangular. Also the index of $\mathbf{S}$ in $\mathbf{T}$ coincides with $\det{}^\star\mathbb{M} = \det{}^\star\mathbb{M}' = \ell m n = \ell' m' n'$. Therefore $(\ell a)(me)(ni) = \ell' m' n'$ implies that $aei = 1$. Also $\ell > 0$, $m > 0$, $n > 0$, $\ell' > 0$, $m' > 0$, $n' > 0$ implies that $a > 0$, $e > 0$, $i > 0$. Therefore $a = e = i = 1$ and $\ell = \ell'$, $m = m'$, $n = n'$. Now considering the subdiagonal entries x', y', z' in ${}^\star\mathbb{M}'$ we have $x' = x + md$, where $0 \leqslant x, x' < m$. Therefore $|x' - x| < m$ and $m|d| = |x' - x|$. So $m|d| < m$ which implies that $d = 0$ as $m > 0$. Similarly we find that $g = h = 0$. Thus $\mathbb{A}$ is the unit matrix and therefore ${}^\star\mathbb{M}$ and ${}^\star\mathbb{M}'$ are identical. A similar proof can be given for arbitrary finite n.

2.3. Classification of canonical matrices

The determinant $\det{}^\star\mathbb{M}$ of the canonical matrix ${}^\star\mathbb{M}$ stated in (2) is given by the product of the diagonal elements. Therefore $\det{}^\star\mathbb{M} = D_n D_{n-1} \cdots D_2 D_1 =: D$. The natural question arises: how many canonical matrices $F(D\,|\,n) \in \mathbb{Z}^+$ of order n are there of given determinant D? The answer to this combinatorical question is quite straightforward. Let $D = D_n D_{n-1} \cdots D_2 D_1$ so that D is factorized into an *ordered* set of n factors, respectively.

Consider now the number of different canonical matrices that possess this ordered set of n factors down the main diagonal. These matrices differ by the different possible values of the independent sub-diagonal elements $E_{j,i}$ which are constrained by $0 \leqslant E_{j,i} \leqslant D_j - 1$ where $i = n, n-1, \ldots, j+1$ and $j = n-1, n-2, \ldots, 2, 1$. In the jth row, counting from below, there are $n - j$ independent elements with each taking all values in the range $0 \leqslant E_{j,i} \leqslant D_j - 1$. The jth row therefore contributes D_j^{n-j} different matrices. The total number of different matrices arising from the given ordered factorization $D_n D_{n-1} \cdots D_2 D_1$ of D is then

$$G(D_n, D_{n-1}, \ldots, D_2, D_1) := (D_n)^0 (D_{n-1})^1 \cdots (D_2)^{n-2} (D_1)^{n-1}. \tag{6}$$

Now let $\pi(D_n, D_{n-1}, \ldots, D_2, D_1)$ denote a *distinct* permutation of the n factors $D_n, D_{n-1}, \ldots, D_2, D_1$. Clearly permuting the factors, permutes the elements down the main diagonal, which leaves the determinant unchanged. Thus, taking all possible distinct sequences of the n factors $D_n, D_{n-1}, \ldots, D_2, D_1$, into account we have the total number of distinct matrices arising from this set of factors of D is

$$H(D_n, D_{n-1}, \ldots, D_2, D_1) := \sum_{\text{all distinct permutations of } \{D_n, D_{n-1}, \ldots, D_2, D_1\}} G(\pi(D_n, D_{n-1}, \ldots, D_2, D_1)). \tag{7}$$

Finally, we sum $H(D_n, D_{n-1}, \ldots, D_2, D_1)$ over all possible distinct un-ordered sets $\{D_j, D_k, \ldots, D_\ell\}$ of admissible factors $D_j : j = 1, 2, \ldots, n$ of D:

$$F(D \mid n) := \sum_{\text{all distinct un-ordered sets of factors } D_n, D_{n-1}, \ldots, D_2, D_1} H(D_n, D_{n-1}, \ldots, D_2, D_1). \tag{8}$$

2.4. Example

We consider the non-trivial example $D = 24$ for $n = 3$. The factors of 24 are: 1, 2, 3, 4, 6, 8, 12, 24. There are six distinct un-ordered sets of three factors whose product is 24:

$$\begin{array}{lll} (a) & \{1, 1, 24\} & H(a) = 601 \\ (b) & \{2, 2, 6\} & H(b) = 104 \\ (c) & \{1, 2, 12\} & H(c) = 498 \\ (d) & \{1, 3, 8\} & H(d) = 348 \\ (e) & \{1, 4, 6\} & H(e) = 302 \\ (f) & \{2, 3, 4\} & H(f) = 162\,. \end{array} \tag{9}$$

For the first two sets there are three distinct permutations, whereas there are six distinct permutations of each or the remaining sets. Simple calulation reveals the values of $H(n)$ with $n = a, b, c, d, e, f$ which are listed above so that $F(D \mid n) = F(24 \mid 3) = 2015$.

2.5. Classification of all subgroups of finite index

The above classification of all canonical matrices of order n according to the value of the determinant D immediately allows the classification of all subgroups $\mathbf{S}$ of index D of a free Abelian group $\mathbf{T}$ of rank n. This follows from the fact that, given an arbitrary but *fixed* basis $\vec{t}$ in $\mathbf{T}$, each and every subgroup $\mathbf{S}$ of finite index D in $\mathbf{T}$ is defined once and once only by one of the $F(D\,|\,n)$ canonical matrices ${}^{\star}\mathbb{M}$ which gives a basis $\vec{s} = \vec{t}\,{}^{\star}\mathbb{M}$ of $\mathbf{S}$. A different choice of basis in $\mathbf{T}$ will result in the same set of subgroups $\mathbf{S}$, but arranged in a different order.

3. Intersection groups

3.1. Intersection group of two subgroups

We begin by considering the case $n = 3$. Let $\mathbf{T}$ denote a translation group of rank 3 and let $\mathbf{T}_1$, $\mathbf{T}_2$ denote two subgroups defined by the canonical integral matrices ${}^{\star}\mathbb{M}_1$ and ${}^{\star}\mathbb{M}_2$, where a canonical integral matrix ${}^{\star}\mathbb{M}$ has the triangular form:

$$ {}^{\star}\mathbb{M} = \begin{bmatrix} \ell & 0 & 0 \\ x & m & 0 \\ y & z & n \end{bmatrix} \qquad 1 \leqslant \ell, m, n \quad 0 \leqslant x \leqslant m-1 \quad 0 \leqslant y, z \leqslant n-1. \tag{10} $$

We claim that the *universal* subgroup $\mathbf{U}(T_1, T_2)$ associated with the two subgroups $\mathbf{T}_1$ and $\mathbf{T}_2$ is defined by the diagonal canonical matrix (where $[a, b]$ denotes the least common multiple of a and b):

$$ {}^{\star}\mathbb{M}(U) = \begin{bmatrix} [\ell_1 m_1 n_1, \ell_2 m_2 n_2] & 0 & 0 \\ 0 & [m_1 n_1, m_2 n_2] & 0 \\ 0 & 0 & [n_1, n_2] \end{bmatrix}. \tag{11} $$

Note that the index $I(\mathbf{U})$ of $\mathbf{U}(T_1, T_2)$ in $\mathbf{T}$, which is the determinant of ${}^{\star}\mathbb{M}(U)$, is given by

$$ I(\mathbf{U}) = [\ell_1 m_1 n_1, \ell_2 m_2 n_2][m_1 n_1, m_2 n_2][n_1, n_2]. \tag{12} $$

Note also that ${}^{\star}\mathbb{M}(U)$ is *independent* of the sub-diagonal entries x_i, y_i, z_i, $i = 1, 2$, in ${}^{\star}\mathbb{M}_i$ and it is in this sense that ${}^{\star}\mathbb{M}(U)$ and the corresponding group $\mathbf{U}(T_1, T_2)$ is *universal*. If it can be shown that $\mathbf{U}(T_1, T_2)$ is a subgroup of $\mathbf{T}_1$ and $\mathbf{T}_2$ then since $\mathbf{U}(T_1, T_2)$ is also a subgroup (proper or improper) of the intersection group $\mathbf{T}_1 \cap \mathbf{T}_2$ of $\mathbf{T}_1$ and $\mathbf{T}_2$, then the index $I(\mathbf{T}_1 \cap \mathbf{T}_2)$ of the intersection group $\mathbf{T}_1 \cap \mathbf{T}_2$ will be bounded above by the index $I(\mathbf{U})$ of $\mathbf{U}(T_1, T_2)$ in $\mathbf{T}$. The diagonal embedding of the group $\mathbf{T}$ in the (external) direct sum of $\mathbf{T}/\mathbf{T}_1 + \mathbf{T}/\mathbf{T}_2$ provides another upper bound on the index $I(\mathbf{T}_1 \cap \mathbf{T}_2)$ of the intersection group $\mathbf{T}_1 \cap \mathbf{T}_2$. The kernel of this embedding is exactly $\mathbf{T}_1 \cap \mathbf{T}_2$ and so $I(\mathbf{T}_1 \cap \mathbf{T}_2)$ is also bounded above by the product of the indices of $\mathbf{T}_1$ and $\mathbf{T}_2$. Therefore we have

$$ I(\mathbf{T}_1 \cap \mathbf{T}_2) \leqslant \min \left\{ I(\mathbf{U}), I(\mathbf{T}_1) * I(\mathbf{T}_2) \right\}. \tag{13} $$

The generalization of the above result to arbitrary finite rank n is obvious.

Rank 1. For this case, there is almost nothing to prove: the intersection group of two subgroups defined by the two 1×1 matrices $[n_1]$ and $[n_2]$ with $n_1, n_2 \in \mathbb{Z}^+$, is defined by the the 1×1 matrix $[[n_1, n_2]]$. Here the lower and upper bounds of the index coincide.

Rank 2. Thus we have

$$ {}^{\star}\mathbb{M}_1 = \begin{bmatrix} m_1 & 0 \\ z_1 & n_1 \end{bmatrix} \tag{14} $$

$$ {}^{\star}\mathbb{M}_2 = \begin{bmatrix} m_2 & 0 \\ z_2 & n_2 \end{bmatrix} \tag{15} $$

and ${}^{\star}\mathbb{M}(U)$, the canonical matrix of the *universal* subgroup, is given by

$$ {}^{\star}\mathbb{M}(U) = \begin{bmatrix} [m_1n_1, m_2n_2] & 0 \\ 0 & [n_1, n_2] \end{bmatrix}. \tag{16} $$

We need to show that $\mathbf{U}$ is a subgroup of both $\mathbf{T}_1$ and $\mathbf{T}_2$. We can do this if we can show that the column vectors of ${}^{\star}\mathbb{M}(U)$ are integral linear combinations of the column vectors of ${}^{\star}\mathbb{M}_1$ and ${}^{\star}\mathbb{M}_2$, respectively. We apply the argument to ${}^{\star}\mathbb{M}_1$ and a similar argument applies to ${}^{\star}\mathbb{M}_2$ as well. It is clear that as n_1 divides $[n_1, n_2]$ then the second column of ${}^{\star}\mathbb{M}(U)$ is an integral multiple of the second column of ${}^{\star}\mathbb{M}_1$. Next, consider the first column of ${}^{\star}\mathbb{M}(U)$. For convenience let $m_3 = [m_1n_1, m_2n_2]$. Now $m_3 = p_1m_1n_1$, where p_1 is integral, so that

$$ \begin{bmatrix} m_3 \\ 0 \end{bmatrix} = p_1n_1 \begin{bmatrix} m_1 \\ z_1 \end{bmatrix} - z_1p_1 \begin{bmatrix} 0 \\ n_1 \end{bmatrix}. \tag{17} $$

So the first column of ${}^{\star}\mathbb{M}(U)$ is expressible as an integral linear combination of the columns of ${}^{\star}\mathbb{M}_1$. Thus $\mathbf{U}$ is a subgroup of $\mathbf{T}_1$ and by a similar argument the same is true for $\mathbf{T}_2$.

Rank n. The proof is by induction on n where $(n \geqslant 1)$. Let $\mathbf{T}$ denote a translation group of rank n and let $\mathbf{T}_1$, $\mathbf{T}_2$ denote two subgroups defined by the canonical integral matrices ${}^{\star}\mathbb{M}_1^n$ and ${}^{\star}\mathbb{M}_2^n$, where a canonical integral matrix has the triangular form as given in (2). Define the subgroup $\mathbf{U}^n(T_1, T_2)$ associated with the two subgroups $\mathbf{T}_1$ and $\mathbf{T}_2$ by the following diagonal canonical matrix ${}^{\star}\mathbb{M}^n(U)$ where the diagonal/off-diagonal elements of ${}^{\star}\mathbb{M}_1^n$ and ${}^{\star}\mathbb{M}_2^n$ are denoted by the subscripted symbols D/E and F/G, respectively:

$$ {}^{\star}\mathbb{M}^n(U) = \begin{bmatrix} \Pi_n & 0 & \cdots & 0 & 0 \\ 0 & \Pi_{n-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Pi_2 & 0 \\ 0 & 0 & \cdots & 0 & \Pi_1 \end{bmatrix} \tag{18} $$

$$ \Pi_j = [D_1D_2\cdots D_j, F_1F_2\cdots F_j] \qquad j = 1, 2, \ldots, n. \tag{19} $$

Let the proposition $P(n)$ be that all the columns of ${}^{\star}\mathbb{M}^n(U)$ are integral linear combinations of the columns of (i) ${}^{\star}\mathbb{M}_1^n$ and (ii) ${}^{\star}\mathbb{M}_2^n$. Assume that $P(n)$ is true for $n = k$, and consider $P(n)$ for $n = k + 1$. Consider ${}^{\star}\mathbb{M}^{k+1}(U)$ given in (18) by setting $n = k + 1$. Observe that apart from the zero entry in the first row of each column, the last k columns of ${}^{\star}\mathbb{M}^{k+1}(U)$ are identical to the k columns of ${}^{\star}\mathbb{M}^k(U)$, and the same is true for the pairs ${}^{\star}\mathbb{M}_1^{k+1}$, ${}^{\star}\mathbb{M}_1^k$ and ${}^{\star}\mathbb{M}_2^{k+1}$, ${}^{\star}\mathbb{M}_2^k$.

By the induction assumption, the k columns of ${}^{\star}\mathbb{M}^k(U)$ are integral linear combinations of (i) ${}^{\star}\mathbb{M}_1^k$ and (ii) ${}^{\star}\mathbb{M}_2^k$. Adding a zero entry at the head of each of the columns of ${}^{\star}\mathbb{M}^k(U)$, ${}^{\star}\mathbb{M}_1^k$ and ${}^{\star}\mathbb{M}_2^k$, it follows that each of the last k columns of ${}^{\star}\mathbb{M}^{k+1}(U)$ is an integral linear combination of the last k columns of (i) ${}^{\star}\mathbb{M}_1^{k+1}$, and (ii) ${}^{\star}\mathbb{M}_2^{k+1}$. It only remains therefore to prove that the first column of ${}^{\star}\mathbb{M}^{k+1}(U)$ is an integral linear combination of the columns of (i) ${}^{\star}\mathbb{M}_1^{k+1}$, and (ii) ${}^{\star}\mathbb{M}_2^{k+1}$. Consider the first column of ${}^{\star}\mathbb{M}_1^{k+1}$. (The same argument applies to the first column of ${}^{\star}\mathbb{M}_2^{k+1}$.) This column is

$(D_{k+1}, E_{k,k+1}, E_{k-1,k+1}, \ldots, E_{2,k+1}, E_{1,k+1})^{\mathrm{T}}$. Consider the expression

$$\frac{1}{P_{k+1}}\begin{bmatrix}\Pi_{k+1}\\0\\0\\\vdots\\0\\0\end{bmatrix} = X_{k+1}\begin{bmatrix}D_{k+1}\\E_{k,k+1}\\E_{k-1,k+1}\\\vdots\\E_{2,k+1}\\E_{1,k+1}\end{bmatrix} + X_k\begin{bmatrix}0\\D_k\\E_{k-1,k}\\\vdots\\E_{2,k}\\E_{1,k}\end{bmatrix} + \cdots + X_1\begin{bmatrix}0\\0\\0\\\vdots\\0\\D_1\end{bmatrix} \tag{20}$$

where the integer P_{k+1} is determined by

$$\Pi_{k+1} = P_{k+1}D_1D_2\cdots D_kD_{k+1}. \tag{21}$$

We need to show that the coefficients X_j, $j = k+1, k, \ldots, 2, 1$ in (20) are all integral. When $j = k+1$, then (20) and (21) imply

$$P_{k+1}D_1D_2\cdots D_kD_{k+1} = \Pi_{k+1} = P_{k+1}X_{k+1}D_{k+1} \tag{22}$$

so that

$$X_{k+1} = D_1D_2\cdots D_k \in \mathbb{Z} \tag{23}$$

is integral. Note that $D_\ell|X_{k+1}$, $\ell = 1, 2, \ldots, k-1, k$. When $j = k$, then (20) implies that

$$0 = X_{k+1}E_{k,k+1} + X_kD_k. \tag{24}$$

Equations (23) and (24) imply that X_k is integral since $D_k|X_{k+1}$. Note that $D_\ell|X_k$, $\ell = 1, 2, \ldots, k-2, k-1$. When $j = k-1$, then (20) implies that

$$0 = X_{k+1}E_{k-1,k+1} + X_kE_{k-1,k} + X_{k-1}D_{k-1} \tag{25}$$

and as $D_{k-1}|X_j$, $j = k, k+1$, then X_{k-1} is integral. Note that $D_\ell|X_{k-1}$, $\ell = 1, 2, \ldots, k-3, k-2$. From the jth row of (20) we obtain

$$0 = X_{k+1}E_{j,k+1} + X_kE_{j,k} + \cdots + X_{j+1}E_{j,j+1} + X_jD_j \qquad j = k+1, k, k-1, \ldots, 2, 1. \tag{26}$$

Now as $D_j|X_\ell$, $\ell = j+1, j+2, \ldots, k, k+1$ then X_j is integral. Therefore, from (20), the first column of $^\star\mathbb{M}^{k+1}(U)$, namely $(\Pi_{k+1}, 0, 0, \ldots, 0)^{\mathrm{T}}$, is an integral linear combination of the $k+1$ columns of $^\star\mathbb{M}_1^{k+1}$. We have therefore shown that all the columns of $^\star\mathbb{M}^{k+1}(U)$ are integral linear combinations of the columns of $^\star\mathbb{M}_1^{k+1}$. By a similar argument all the columns of $^\star\mathbb{M}^{k+1}(U)$ are integral linear combinations of the columns of $^\star\mathbb{M}_2^{k+1}$ too. Therefore if $P(k)$ is true then $P(k+1)$ is true also. We have established that $P(n)$ is true for $n = 2$ (and $P(1)$ is trivially true) so that $P(n)$ is true for all $n \geqslant 1$. Note that we cannot appeal to the truth of $P(1)$ as the starting point instead of $P(2)$ since for $n = 1$, $^\star\mathbb{M}_1^1$, $^\star\mathbb{M}_2^1$, $^\star\mathbb{M}^1(U)$ lose their triangular structure which is essential to the argument.

Corollaries. It follows that $\left(^\star\mathbb{M}_1^n\right)^{-1}\ {}^\star\mathbb{M}^n(U)$ and $\left(^\star\mathbb{M}_2^n\right)^{-1}\ {}^\star\mathbb{M}^n(U)$ are integral matrices which define the basis vectors of $^\star\mathbb{M}^n(U)$ as integral linear combinations of the basis vectors of $\mathbf{T}_1$ and $\mathbf{T}_2$ respectively. If $\mathbf{T}_2 = \mathbf{T}$ then all subgroups $\mathbf{T}_1$, with the diagonal elements fixed but with varying off-diagonal elements, are supergroups of $\mathbf{U}$ where $^\star\mathbb{M}(U) = \mathrm{diag}(\Pi_n, \Pi_{n-1}, \ldots, \Pi_2, \Pi_1)$ where $\Pi_j = D_1D_2\cdots D_j$ with $j = 1, 2, \ldots, n$.

3.2. Construction of intersection groups

Next we describe an algorithm for constructing the canonical matrix ${}^{\star}\mathbb{M}^n(T_1 \cap T_2)$ of the intersection group $\mathbf{T}_1 \cap \mathbf{T}_2$ of two subgroups $\mathbf{T}_1$ and $\mathbf{T}_2$ of a free Abelian group $\mathbf{T}$ of rank n, where the subgroups are defined by canonical matrices ${}^{\star}\mathbb{M}_1^n$, ${}^{\star}\mathbb{M}_2^n$ respectively. We consider in detail the cases $n = 1, 2, 3$ which are sufficient to show how the algorithm generalizes to arbitrary finite higher values of n.

Rank 1. The intersection group $\mathbf{T}_1 \cap \mathbf{T}_2$ of two subgroups $\mathbf{T}_1$ and $\mathbf{T}_2$ defined respectively by the two 1×1 matrices $[n_1]$ and $[n_2]$ with $n_1, n_2 \in \mathbb{Z}^+$, is defined by the the 1×1 matrix $[[n_1, n_2]]$.

Rank 2. We are given two canonical matrices ${}^{\star}\mathbb{M}_1^2$ and ${}^{\star}\mathbb{M}_2^2$:

$$ {}^{\star}\mathbb{M}_1^2 = \begin{bmatrix} m_1 & 0 \\ z_1 & n_1 \end{bmatrix} \tag{27} $$

$$ {}^{\star}\mathbb{M}_2^2 = \begin{bmatrix} m_2 & 0 \\ z_2 & n_2 \end{bmatrix} \tag{28} $$

where $1 \leqslant m_i, n_i$, and $0 \leqslant z_i \leqslant n_i - 1$, $i = 1, 2$, and we wish to find the canonical matrix ${}^{\star}\mathbb{M}^2(T_1 \cap T_2)$ of the intersection group $\mathbf{T}_1 \cap \mathbf{T}_2$. Let

$$ {}^{\star}\mathbb{M}^2(T_1 \cap T_2) = \begin{bmatrix} m_3 & 0 \\ z_3 & n_3 \end{bmatrix} \tag{29} $$

where $1 \leqslant m_3, n_3$, and $0 \leqslant z_3 \leqslant n_3 - 1$. Consider the last column of ${}^{\star}\mathbb{M}_1^2$, ${}^{\star}\mathbb{M}_2^2$ and ${}^{\star}\mathbb{M}(T_1 \cap T_2)$ in which the entry in the first row is zero so that these columns represent elements in a subgroup of $\mathbf{T}$ of rank 1. It follows immediately from the discussion above for Rank 1 that $n_3 = [n_1, n_2]$. Consider next the first column of ${}^{\star}\mathbb{M}_1^2$, ${}^{\star}\mathbb{M}_2^2$ and ${}^{\star}\mathbb{M}(T_1 \cap T_2)$. Take the special case $z_1 = z_2 = 0$ so that these columns represent elements in another subgroup of $\mathbf{T}$ of rank 1. As above this implies that $m_3 = [m_1, m_2]$ and $z_3 = 0$ so that the index $I(\mathbf{T}_1 \cap \mathbf{T}_2) = m_3 n_3 = [m_1, m_2][n_1, n_2]$. Note that this index divides the index $[m_1 n_1, m_2 n_2][n_1, n_2]$ of the *universal* subgroup $\mathbf{U}(T_1, T_2)$ as required. Finally consider the general case where not both z_1 and z_2 are zero. The first column of ${}^{\star}\mathbb{M}(T_1 \cap T_2)$ is by definition an integral linear combination of (i) the columns of ${}^{\star}\mathbb{M}_1^2$ and (ii) the columns of ${}^{\star}\mathbb{M}_2^2$. So

$$ \begin{aligned} \begin{bmatrix} m_3 \\ z_3 \end{bmatrix} &= \frac{m_3}{m_1}\begin{bmatrix} m_1 \\ z_1 \end{bmatrix} + p_1 \begin{bmatrix} 0 \\ n_1 \end{bmatrix} \qquad m_3/m_1 \in \mathbb{Z} \quad p_1 \in \mathbb{Z} \\ \begin{bmatrix} m_3 \\ z_3 \end{bmatrix} &= \frac{m_3}{m_2}\begin{bmatrix} m_2 \\ z_2 \end{bmatrix} + p_2 \begin{bmatrix} 0 \\ n_2 \end{bmatrix} \qquad m_3/m_2 \in \mathbb{Z} \quad p_2 \in \mathbb{Z} \end{aligned} \tag{30} $$

and $0 \leqslant z_3 \leqslant n_3 - 1 = [n_1, n_2] - 1$. Or equivalently n_1 divides $(z_3 - (m_3/m_1)z_1)$ and n_2 divides $(z_3 - (m_3/m_2)z_2)$ for some z_3 in $0 \leqslant z_3 \leqslant n_3 - 1 = [n_1, n_2] - 1$. A non-trivial consequence of the above is that m_3 must be a common multiple of both m_1 and m_2 and therefore is bounded below by $[m_1, m_2]$. If a solution for z_3 in (30) exists when $m_3 = [m_1, m_2]$ then the lower bound on the index $I(\mathbf{T}_1 \cap \mathbf{T}_2) = m_3 n_3 = [m_1, m_2][n_1, n_2]$ is attained. From equation (13) it follows that m_3 is bounded above by $m_3^{\max}$, where

$$ m_3^{\max} = \min\left\{[m_1 n_1, m_2 n_2], (m_1 n_1)(m_2 n_2)/[n_1, n_2]\right\}. \tag{31} $$

A solution for z_3 in (30) must exist for $m_3 = p[m_1, m_2]$, for some integral p, $p = 1, 2, \ldots, p_{\max}$, where

$$ p_{\max} = m_3^{\max}/[m_1, m_2] = \min\left\{[m_1 n_1, m_2 n_2]/[m_1, m_2], (m_1, m_2)(n_1, n_2)\right\} \tag{32} $$

using $ab = (a, b)[a, b]$ where (a, b) denotes the highest common factor of a and b. Furthermore, the uniqueness of $\mathbf{T}_1 \cap \mathbf{T}_2$ implies that if a solution for z_3 is found for $p = p'$ then no other solution for z_3 exists for this value p of p. As p is increased from unity to $p_{\max}$ either no solution for z_3 exists or precisely one solution exists (for $p = p'$) whereupon the search stops and the intersection matrix is found and has determinant $p'[m_1, m_2][n_1, n_2]$. The upper bound on the index is attained when $p' = p_{\max}$ so that

$$I(\mathbf{T}_1 \cap \mathbf{T}_2) = \min\left\{[m_1 n_1, m_2 n_2][n_1, n_2], (m_1 n_1)(m_2 n_2)\right\}. \tag{33}$$

Rank 3. We are given two canonical matrices ${}^{\star}\mathbb{M}_1^3$ and ${}^{\star}\mathbb{M}_2^3$:

$${}^{\star}\mathbb{M}_1^3 = \begin{bmatrix} \ell_1 & 0 & 0 \\ x_1 & m_1 & 0 \\ y_1 & z_1 & n_1 \end{bmatrix} \tag{34}$$

$${}^{\star}\mathbb{M}_2^3 = \begin{bmatrix} \ell_2 & 0 & 0 \\ x_2 & m_2 & 0 \\ y_2 & z_2 & n_2 \end{bmatrix} \tag{35}$$

where $1 \leqslant \ell_i, m_i, n_i$, $0 \leqslant x_i \leqslant m_i - 1$, $0 \leqslant y_i, z_i \leqslant n_i - 1$, $i = 1, 2$, and we wish to find the canonical matrix ${}^{\star}\mathbb{M}^3(T_1 \cap T_2)$ of the intersection group $\mathbf{T}_1 \cap \mathbf{T}_2$. Let

$${}^{\star}\mathbb{M}^3(T_1 \cap T_2) = \begin{bmatrix} \ell_3 & 0 & 0 \\ x_3 & m_3 & 0 \\ y_3 & z_3 & n_3 \end{bmatrix} \tag{36}$$

where $1 \leqslant \ell_3, m_3, n_3$, $0 \leqslant x_3 \leqslant m_3 - 1$, $0 \leqslant y_3, z_3 \leqslant n_3 - 1$. The second and third columns of ${}^{\star}\mathbb{M}^3(T_1 \cap T_2)$ are found as described in the rank 2 case above. It remains to describe how the elements of the first column are found. The first column of ${}^{\star}\mathbb{M}^3(T_1 \cap T_2)$ is by definition an integral linear combination of (i) the columns of ${}^{\star}\mathbb{M}_1^3$ and (ii) the columns of ${}^{\star}\mathbb{M}_2^3$. So

$$\begin{aligned} \begin{bmatrix} \ell_3 \\ x_3 \\ y_3 \end{bmatrix} &= \frac{\ell_3}{\ell_1} \begin{bmatrix} \ell_1 \\ x_1 \\ y_1 \end{bmatrix} + q_1 \begin{bmatrix} 0 \\ m_1 \\ z_1 \end{bmatrix} + r_1 \begin{bmatrix} 0 \\ 0 \\ n_1 \end{bmatrix} \qquad (\ell_3/\ell_1), q_1, r_1 \in \mathbb{Z} \\ \begin{bmatrix} \ell_3 \\ x_3 \\ y_3 \end{bmatrix} &= \frac{\ell_3}{\ell_2} \begin{bmatrix} \ell_2 \\ x_2 \\ y_2 \end{bmatrix} + q_2 \begin{bmatrix} 0 \\ m_2 \\ z_2 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ n_2 \end{bmatrix} \qquad (\ell_3/\ell_2), q_2, r_2 \in \mathbb{Z}. \end{aligned} \tag{37}$$

Finally, $0 \leqslant x_3 \leqslant m_3 - 1 = p'[m_1, m_2] - 1$, $0 \leqslant y_3 \leqslant n_3 - 1 = [n_1, n_2] - 1$, where the value of p' is found from the calculation of z_3 (see rank 2 above). A non-trivial consequence of the above is that ℓ_3 must be a common multiple of both ℓ_1 and ℓ_2 and therefore is bounded below by $[\ell_1, \ell_2]$. If a solution for x_3 and y_3 in (37) exists when $\ell_3 = [\ell_1, \ell_2]$ and $p' = 1$ then the lower bound on the index $I(\mathbf{T}_1 \cap \mathbf{T}_2) = \ell_3 m_3 n_3 = [\ell_1, \ell_2][m_1, m_2][n_1, n_2]$ is attained. From equation (13) it follows that ℓ_3 is bounded above by $\ell_3^{\max}$, where

$$\ell_3^{\max} = \min\left\{[\ell_1 m_1 n_1, \ell_2 m_2 n_2], (\ell_1 m_1 n_1)(\ell_2 m_2 n_2)/(p'[m_1, m_2][n_1, n_2])\right\}. \tag{38}$$

A solution for x_3 and y_3 in (37) must exist for $\ell_3 = q[\ell_1, \ell_2]$, for some integral q, $q = 1, 2, \ldots, q_{\max}$, where

$$q_{\max} = \ell_3^{\max}/[\ell_1, \ell_2] = \min\left\{[\ell_1 m_1 n_1, \ell_2 m_2 n_2]/[\ell_1, \ell_2], (\ell_1, \ell_2)(m_1, m_2)(n_1, n_2)/p'\right\}. \tag{39}$$

Furthermore, the uniqueness of $\mathbf{T}_1 \cap \mathbf{T}_2$ implies that if a solution for x_3 and y_3 is found for $q = q'$ then no other solution for x_3 and y_3 exists for this value q' of q. As q is increased from unity to $q_{\max}$ either no solution for x_3 and y_3 exists or precisely one solution exists (for $q = q'$) whereupon the search stops and the intersection matrix is found and has determinant $q'[\ell_1, \ell_2]p'[m_1, m_2][n_1, n_2]$. The upper bound on the index is attained when $p' = p_{\max}$ and $q' = q_{\max}$ so that

$$I(\mathbf{T}_1 \cap \mathbf{T}_2) = \min\left\{[\ell_1 m_1 n_1, \ell_2 m_2 n_2][m_1 n_1, m_2 n_2][n_1, n_2], (\ell_1 m_1 n_1)(\ell_2 m_2 n_2)\right\}. \tag{40}$$

Rank n. The generalization of the above to higher values of n, namely ($n > 3$), is natural and straightforward, and we leave the interested reader to carry this out for himself.

4. Union groups

4.1. Union group of two subgroups

We define the union group $\mathbf{T}_3$ of two subgroups $\mathbf{T}_1$ and $\mathbf{T}_2$ of $\mathbf{T}$ as that subgroup of $\mathbf{T}$ generated by $\mathbf{T}_1$ and $\mathbf{T}_2$. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be defined by canonical matrices ${}^{\star}\mathbb{M}_1$ and ${}^{\star}\mathbb{M}_2$ of the form given by equation (2). We need to find the $n \times n$ canonical matrix ${}^{\star}\mathbb{M}_3$ which determines $\mathbf{T}_3$. This is done by employing the Euclidean algorithm, as in subsection 2.1, to reduce the $n \times 2n$ rectangular matrix $\left[{}^{\star}\mathbb{M}_1, {}^{\star}\mathbb{M}_2\right]$ by elementary column operations to the form $\left[{}^{\star}\mathbb{M}_3, \emptyset\right]$, where Ø here denotes the $n \times n$ zero matrix.

For simplicity we illustrate the procedure for the case $n = 2$. For higher values of n one simply repeats the procedure for $n = 2$. We use elementary column operations as in Section 2.1 on the 2×4 matrix $\left[{}^{\star}\mathbb{M}_1, {}^{\star}\mathbb{M}_2\right]$:

$$\begin{bmatrix} \ell_1 & 0 & \ell_2 & 0 \\ x_1 & m_1 & x_2 & m_2 \end{bmatrix}. \tag{41}$$

This is column equivalent to:

$$\begin{bmatrix} \ell_1 & \ell_2 & 0 & 0 \\ x_1 & x_2 & m_1 & m_2 \end{bmatrix}. \tag{42}$$

By using the Euclidean algorithm to find the greatest common divisor (a, b) of two integers a, b, the first two columns of the above are column equivalent to

$$\begin{bmatrix} (\ell_1, \ell_2) & 0 \\ x_1' & x_2' \end{bmatrix} \tag{43}$$

and the third and fourth columns are column equivalent to

$$\begin{bmatrix} 0 & 0 \\ (m_1, m_2) & 0 \end{bmatrix}. \tag{44}$$

Therefore $\left[{}^{\star}\mathbb{M}_1, {}^{\star}\mathbb{M}_2\right]$ is column equivalent to

$$\begin{bmatrix} (\ell_1, \ell_2) & 0 & 0 & 0 \\ x_1' & x_2' & (m_1, m_2) & 0 \end{bmatrix}. \tag{45}$$

Now use the Euclidean algorithm again on the second and third columns which shows that $\left[{}^{\star}\mathbb{M}_1, {}^{\star}\mathbb{M}_2\right]$ is column equivalent to

$$\begin{bmatrix} (\ell_1, \ell_2) & 0 & 0 & 0 \\ x_1' & (x_2', (m_1, m_2)) & 0 & 0 \end{bmatrix}. \tag{46}$$

Note that $(x_2', (m_1, m_2))$ divides (m_1, m_2). Finally reduce the first two columns to canonical form by elementary column operations:

$$\begin{bmatrix} (\ell_1, \ell_2) & 0 & 0 & 0 \\ x_3' & (x_2', (m_1, m_2)) & 0 & 0 \end{bmatrix} \tag{47}$$

where $0 \leqslant x_3' \leqslant (x_2', (m_1, m_2)) - 1$. The canonical form $^\star\mathbb{M}_3$ is then

$$\begin{bmatrix} (\ell_1, \ell_2) & 0 \\ x_3' & (x_2', (m_1, m_2)) \end{bmatrix}. \tag{48}$$

A corollary of the above is that $1 \leqslant \det {}^\star\mathbb{M}_3 \leqslant (\ell_1, \ell_2)(m_1, m_2)$.

The procedure for rank n is essentially a repetition of the above, starting with the first column of $^\star\mathbb{M}_1$ and the first column of $^\star\mathbb{M}_2$. We find that $1 \leqslant \det {}^\star\mathbb{M}_3 \leqslant (D_n, F_n)(D_{n-1}, F_{n-1}) \cdots (D_1, F_1)$. As a final remark we note that it is not possible to find a greater lower bound than unity on $\det {}^\star\mathbb{M}_3$ which depends only on the diagonal elements of $^\star\mathbb{M}_1$ and $^\star\mathbb{M}_2$. This can be seen by considering

$$^\star\mathbb{M}_1 = \begin{bmatrix} 1 & 0 \\ x & m \end{bmatrix} \qquad ^\star\mathbb{M}_2 = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}. \tag{49}$$

If x is co-prime to m then $^\star\mathbb{M}_3$ is the two-dimensional unit matrix.

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